Forward Kinematics: Product of Exponentials Formula

CS 6341 Robotics
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Twist

- Let's combine angular velocity and linear velocity into a 6D vector called twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

- Twist can be defined in fixed frame or body frame

Spatial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$ Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$

Relationship between Spatial Twist and Body Twist

- For angular velocity $\omega_s = R\omega_b$

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

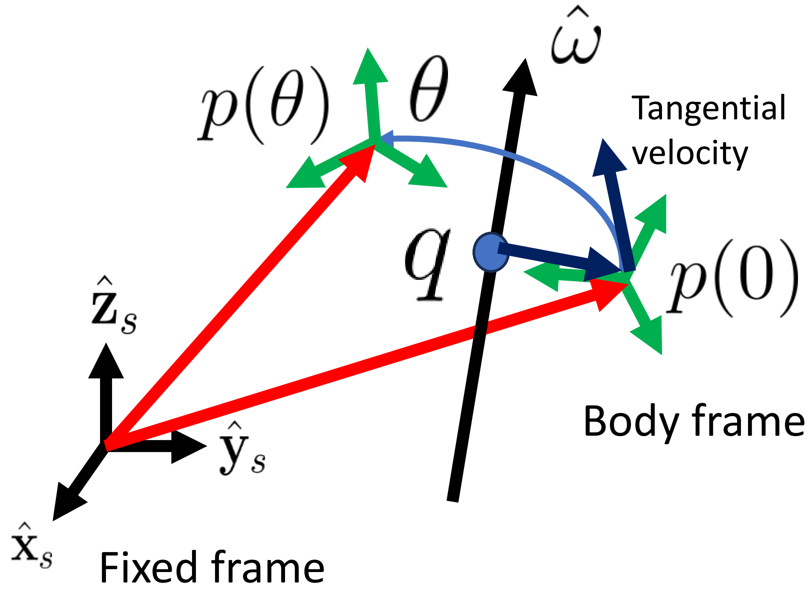
- For linear velocity

$$v_s = -\omega_s \times q_s + v_s^0$$

$$v_b = -\omega_b \times q_b + v_b^0$$

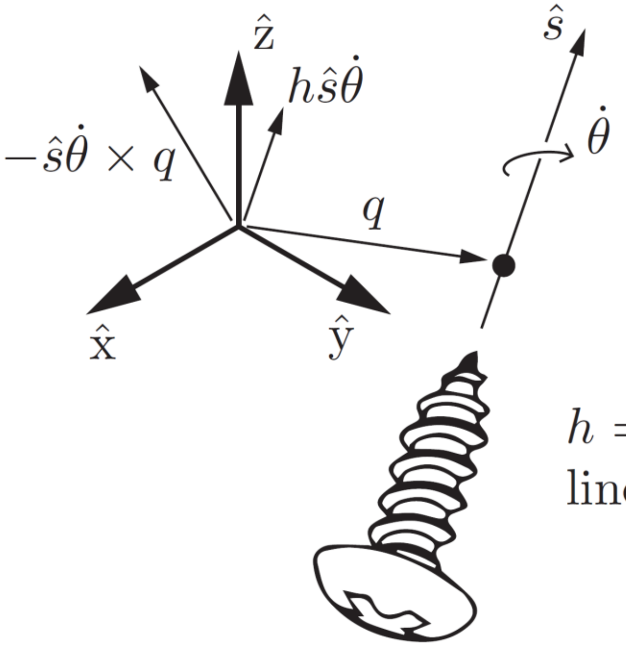
An additional linear velocity if exists

$$q_s = Rq_b + p \qquad v_s^0 = Rv_b^0$$



The Screw Interpretation of a Twist

- Screw axis: motion of a screw
  - Rotating about the axis while translating along the axis



h = pitch = linear speed/angular speed

Screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$

$q \in \mathbb{R}^3$ is a point on the axis (any point is fine)

Twist about S with angular velocity $\dot{\theta}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

The Screw Interpretation of a Twist

- For any twist $\mathcal{V} = (\omega, v) \quad \omega \neq 0$
- These exists $\{q, \hat{s}, h\} \quad \dot{\theta}$

$$\omega = \hat{s}\dot{\theta}$$

$$v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$$

$$\hat{s} = \omega / \|\omega\| \qquad \dot{\theta} = \|\omega\| \qquad h = \hat{\omega}^{\mathrm{T}} v / \dot{\theta}$$

portion of v parallel to the screw axis

$$-\hat{s}\dot{\theta} \times q = v - h\hat{s}\dot{\theta}$$

provides the portion of v orthogonal to the screw axis
(choose q based on this term)

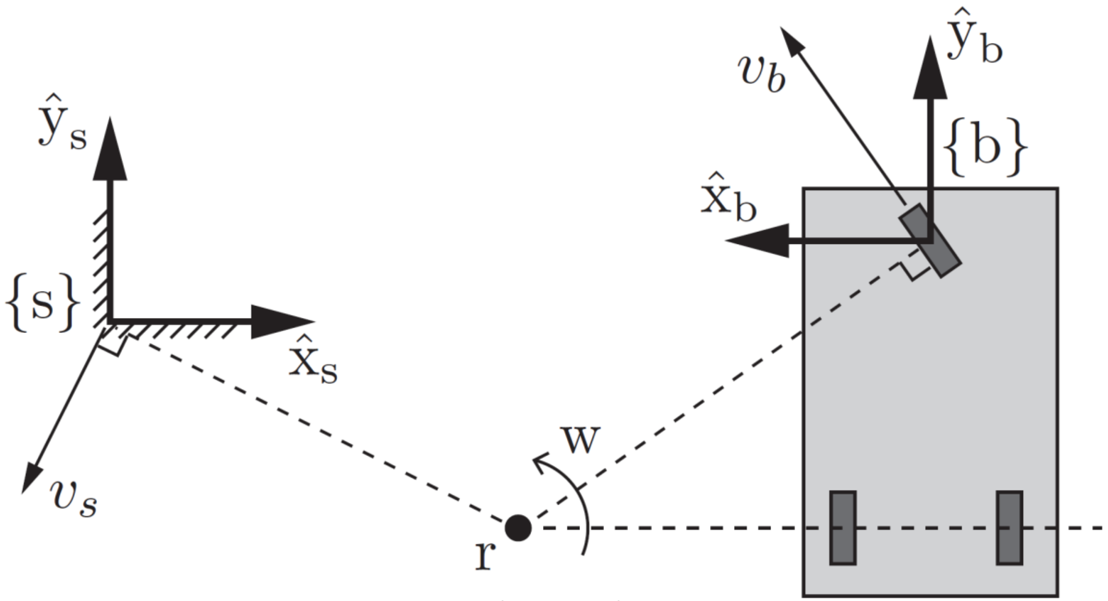
If $\omega = 0$

$$\hat{s} = v / \|v\|$$

h = pitch = linear speed/angular speed infinity

$\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$

Twists Example



Rotation axis

Top view of a car

- Pure Angular velocity $\mathrm{w} = 2 \text{ rad/s}$

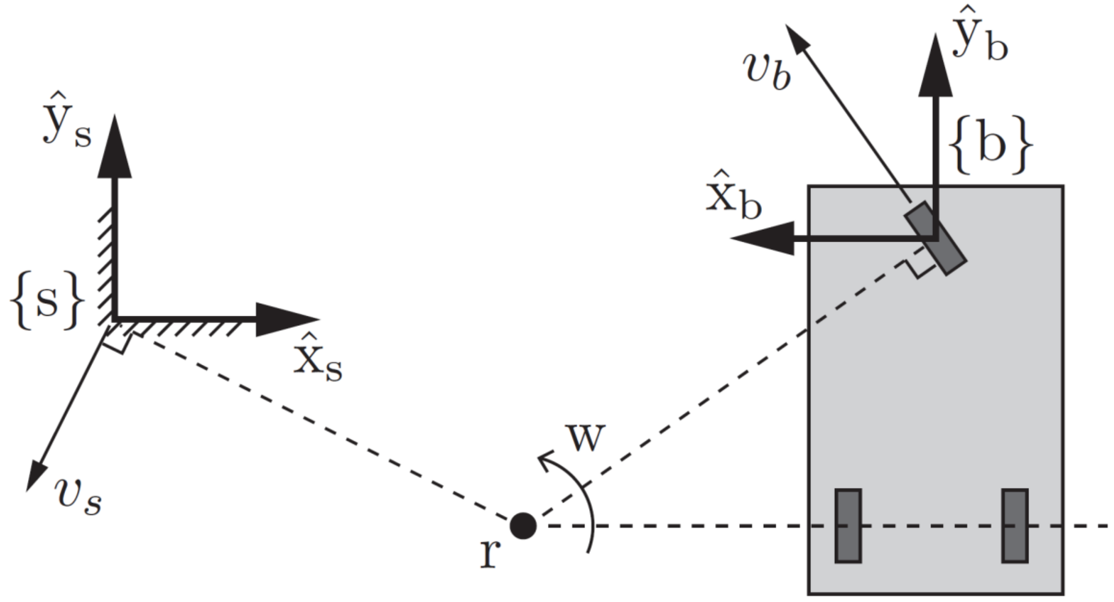
$r_s = (2, -1, 0) \quad r_b = (2, -1.4, 0)$

$\omega_s = (0, 0, 2) \quad \omega_b = (0, 0, -2)$

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the linear velocities? $v_s \quad v_b$

Twists Example



Top view of a car

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Linear velocity of the car

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \qquad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

Cross Product

- Matrix notation

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

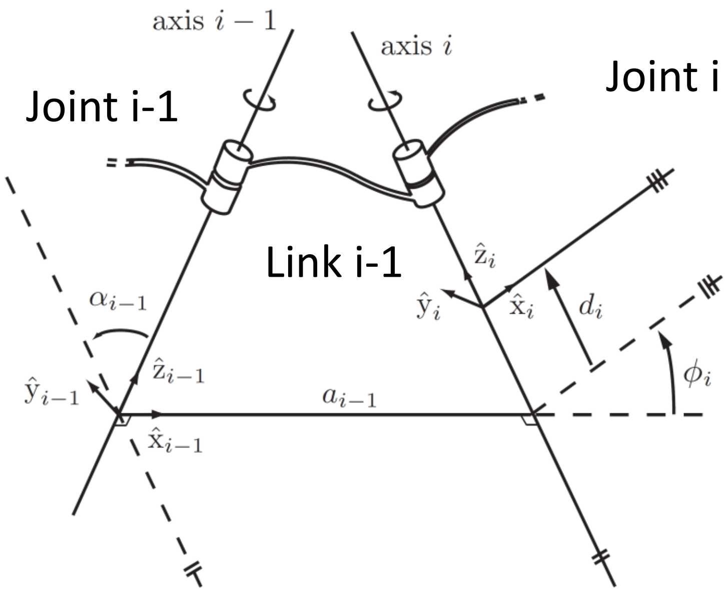
$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}, \end{aligned}$$

https://en.wikipedia.org/wiki/Cross_product

Forward Kinematics with D-H Parameters

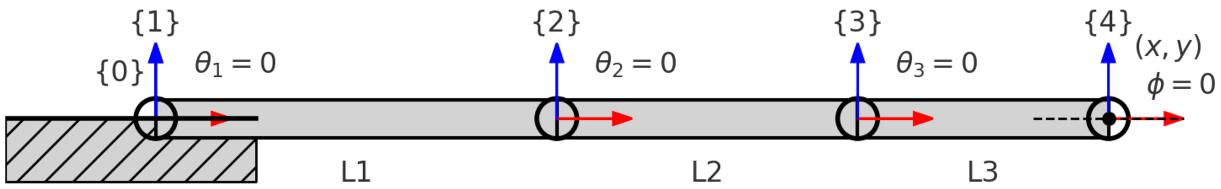
- Link frame transformation

$$T_{i-1,i} \quad = \quad \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$



$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2)\cdots T_{n-1,n}(\theta_n)$$

Forward Kinematics: Product of Exponentials Formula



Forward kinematics of a 3R planar open chain.

- A different approach

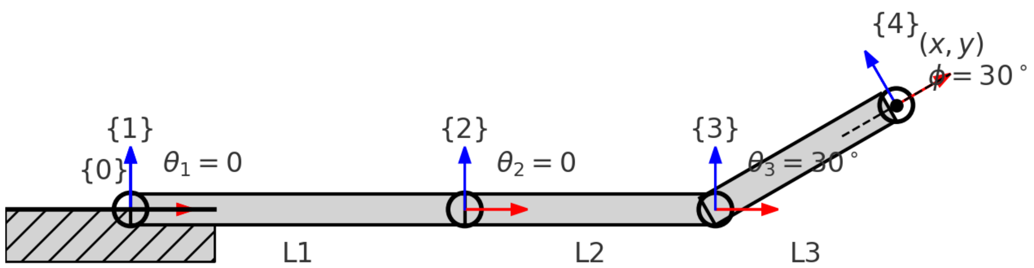
- Define M to the position and orientation of frame {4} when all the joint angles are zeros ("home" or "zero" position of the robot)

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

- Consider each revolute joint as a zero-pitch (no additional linear velocity) screw-axis expressed in the {0} frame (fixed frame)

Planar 3R Arm with $\theta_3 = 30^\circ$, $\theta_1 = \theta_2 = 0$



Forward kinematics of a 3R planar open chain.

For joint 3

Spatial twist

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} \qquad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

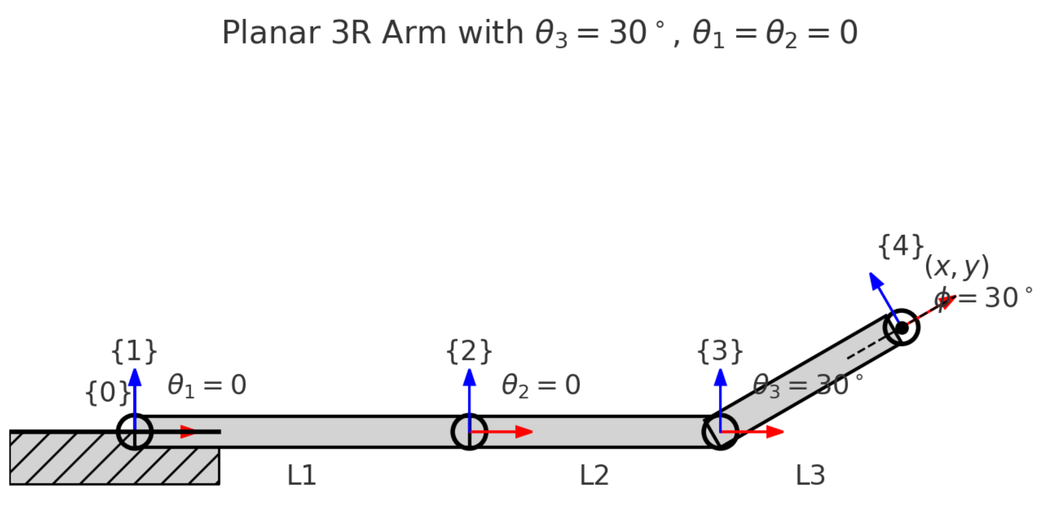
$$v_3 = -\omega_3 \times q_3$$

$$q_3 = (L_1 + L_2, 0, 0)$$

$$v_3 = \begin{bmatrix} 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

Forward Kinematics



Forward kinematics of a 3R planar open chain.

- Consider each revolute joint as a zero-pitch (no additional linear velocity) screw-axis expressed in the {0} frame (fixed frame)

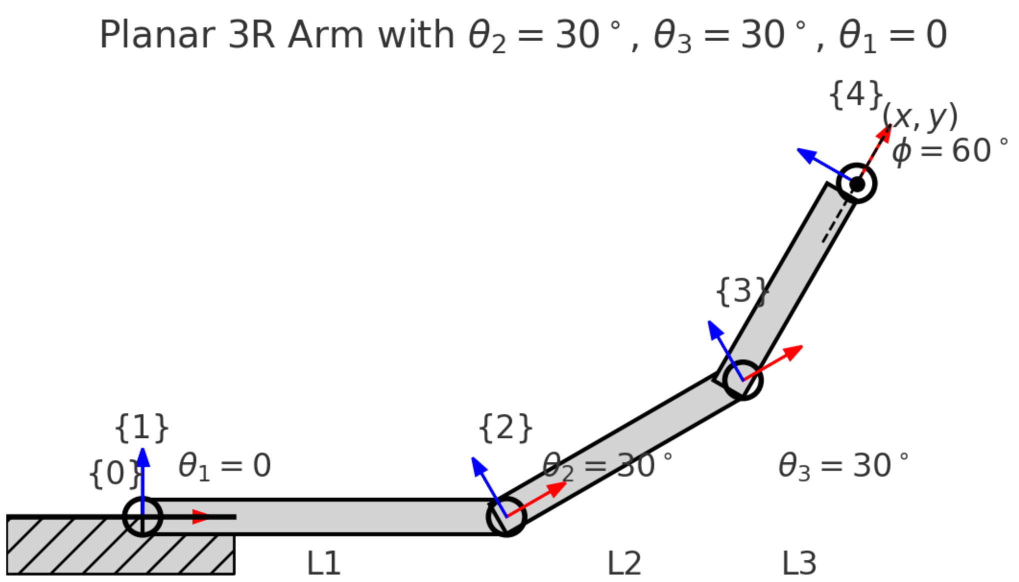
$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0)$$

$$T(\theta) = e^{[\mathcal{S}]\theta} = \begin{bmatrix} I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 & (I\theta + (1 - \cos\theta)[\hat{\omega}] + (\theta - \sin\theta)[\hat{\omega}]^2)v \\ 0 & 1 \end{bmatrix}$$

Exponential Coordinates of Rigid-Body Motions

Forward Kinematics



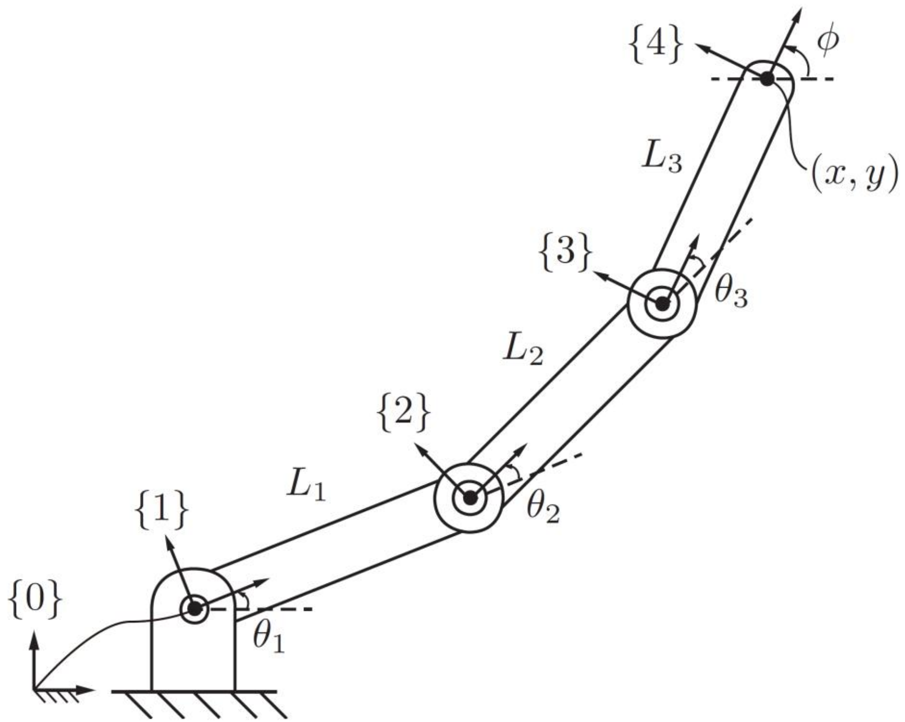
Forward kinematics of a 3R planar open chain.

- Consider each revolute joint as a zero-pitch (no additional linear velocity) screw-axis expressed in the {0} frame (fixed frame)

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0)$$

Forward Kinematics

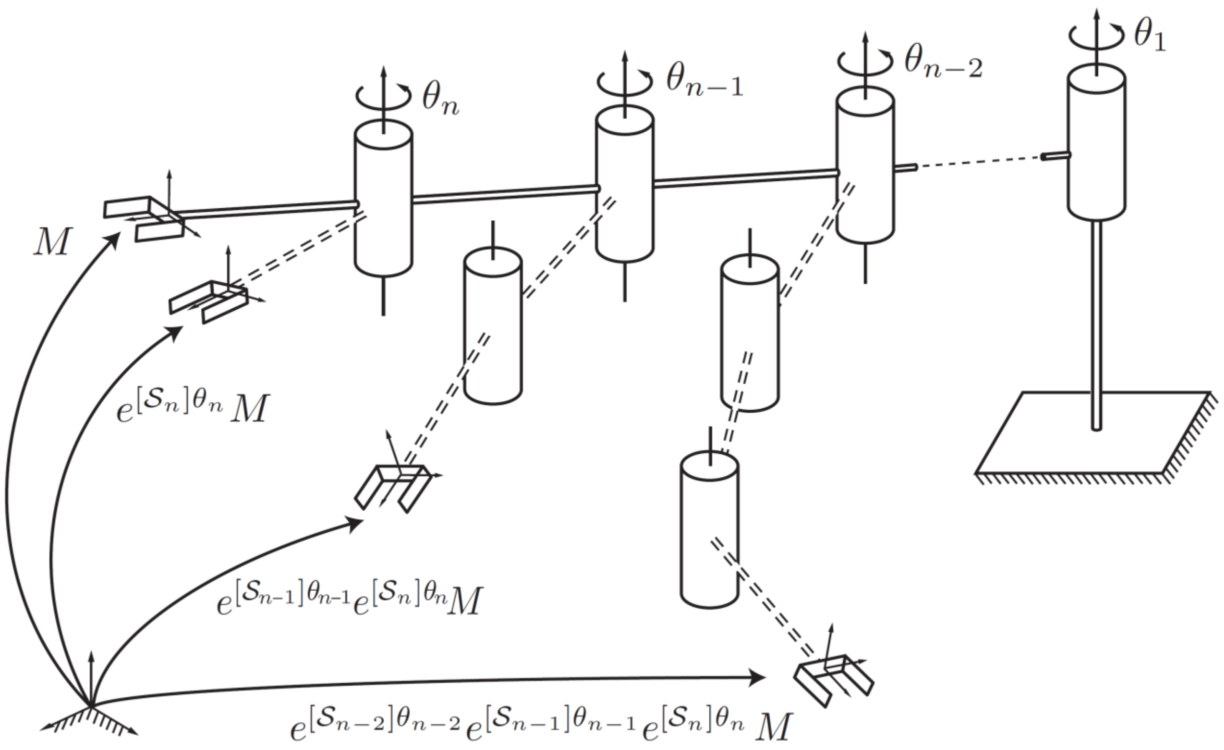


$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

a product of matrix exponentials
(does not use any frame references, only {0} and M)

Product of Exponentials Formula



- Each link apply a screw motion to all the outward links
- Base frame {s}
- End-effector frame {b}

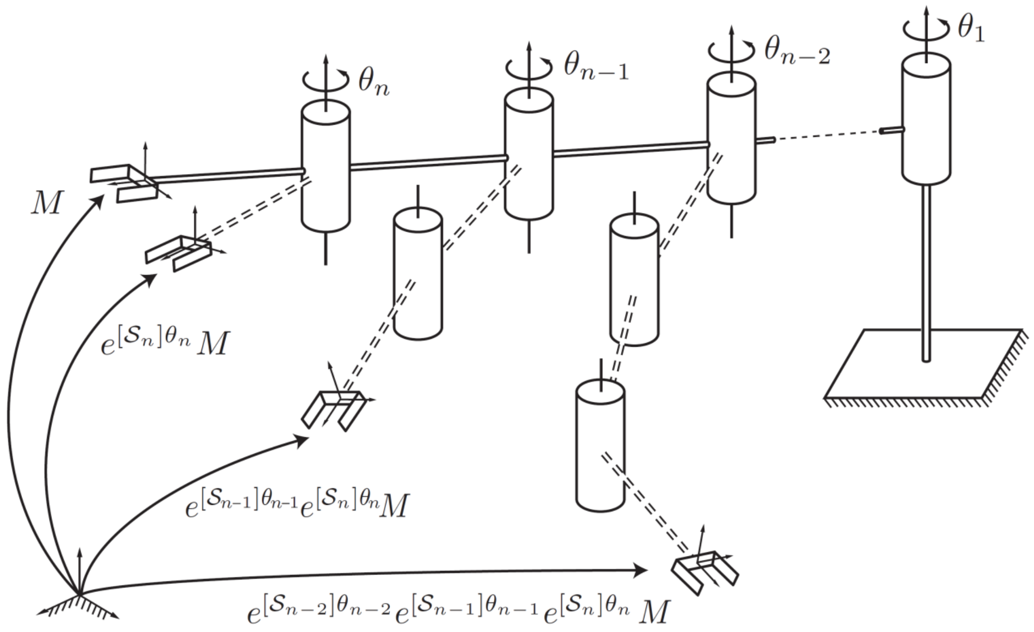
$$M \in SE(3)$$

{b} in {s} when all the joint values are zeros

$$T = e^{[\mathcal{S}_n]\theta_n} M$$

{b} in {s} when joint n with value θ_n

Product of Exponentials Formula

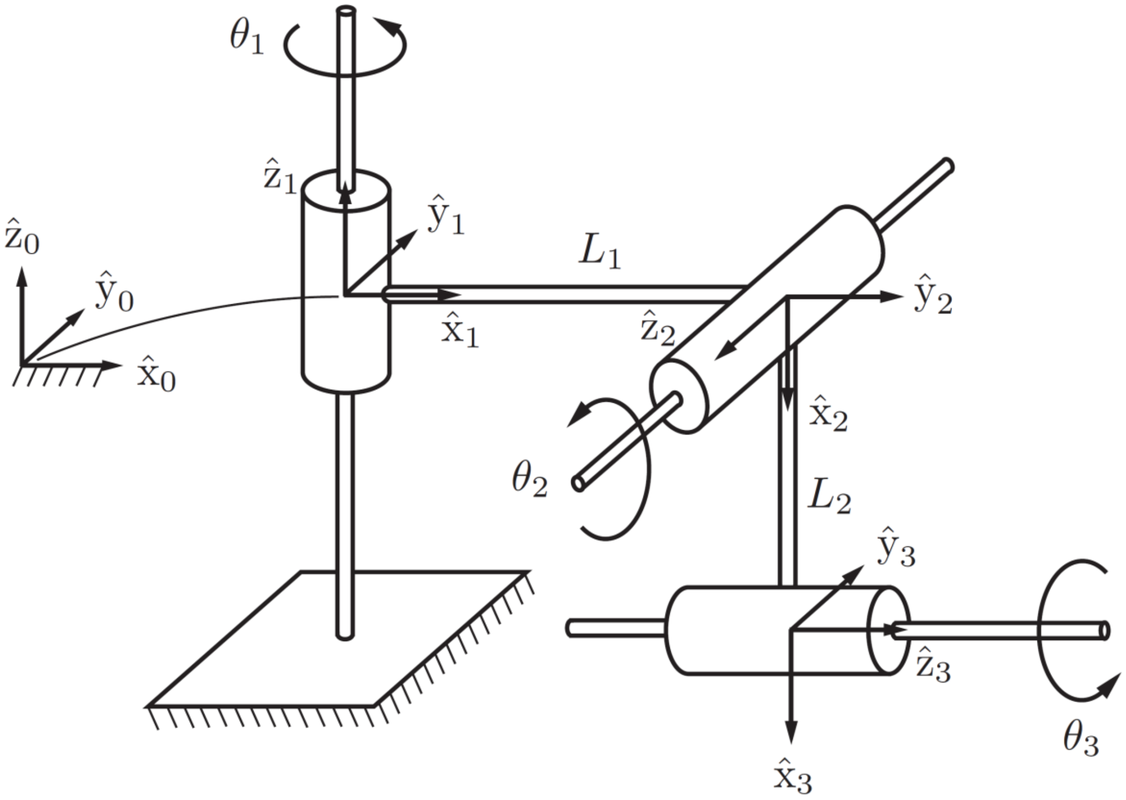


$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Joint values $(\theta_1, \dots, \theta_n)$

- Space form of the product of exponentials formula
- Unlike D-H representation, no link reference frames need to be defined

Product of Exponentials Formula



A 3R spatial open chain

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{S}_1 = (\omega_1, v_1) \quad \omega_1 = (0,0,1) \quad v_1 = (0,0,0)$$

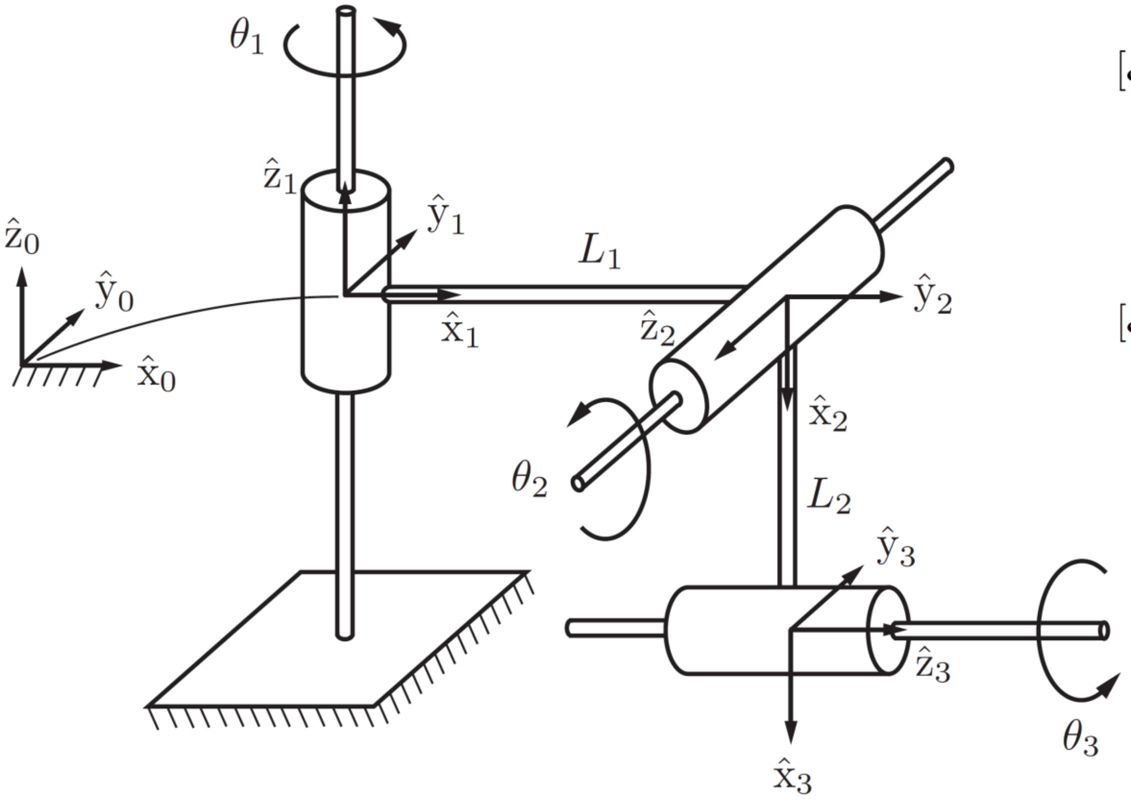
$$\omega_2 = (0,-1,0) \quad q_2 = (L_1, 0, 0)$$

$$v_2 = -\omega_2 \times q_2 = (0,0,-L_1)$$

$$\omega_3 = (1,0,0) \quad q_3 = (0,0,-L_2)$$

$$v_3 = -\omega_3 \times q_3 = (0,-L_2,0)$$

Product of Exponentials Formula



A 3R spatial open chain

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Can we use body twist?

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b \qquad [\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Additional results about twist (Lynch & Park 3.3.2)

$$\begin{aligned} \dot{R}R^{-1} &= [\omega_s] \\ R^{-1}\dot{R} &= [\omega_b] \end{aligned}$$

$$[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned}$$

$$[\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} = T^{-1}\dot{T}$$

$$[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

Screw Axes in the End-Effector Frame

- Proposition 3.10 (Lynch & Park) $e^{M^{-1}PM} = M^{-1}e^{P}M \quad Me^{M^{-1}PM} = e^{P}M$
- PoE formula

$$[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$$

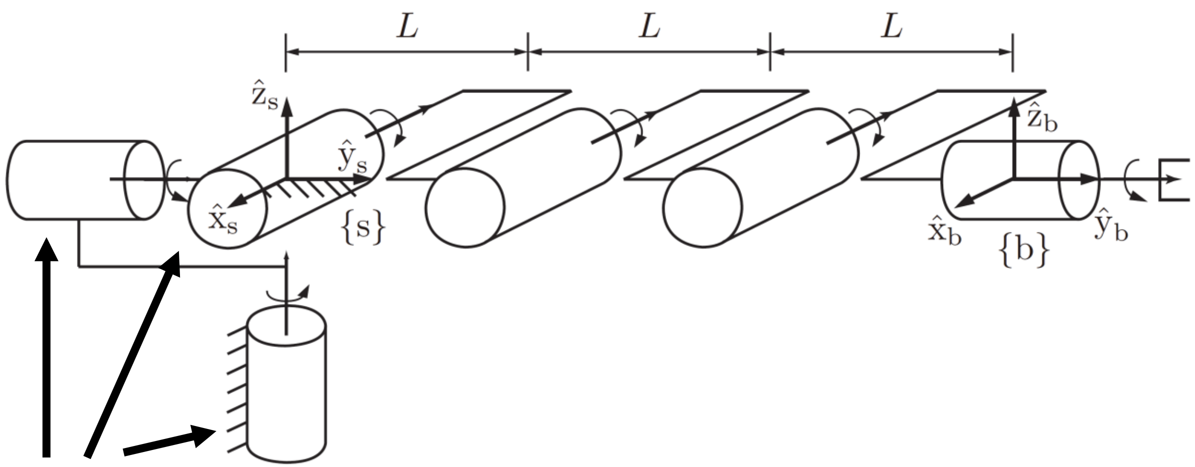
$$\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i, \; i = 1, \ldots, n$$

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots \boxed{e^{[\mathcal{S}_n]\theta_n} M} \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n} \end{aligned}$$

Body form of the product of exponentials formula

Screw axes B_i in the end-effector (body) frame when the robot is at its zero position

Screw Axes in the End-Effector Frame



These 3 joints are at the same location

PoE forward kinematics for the 6R open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

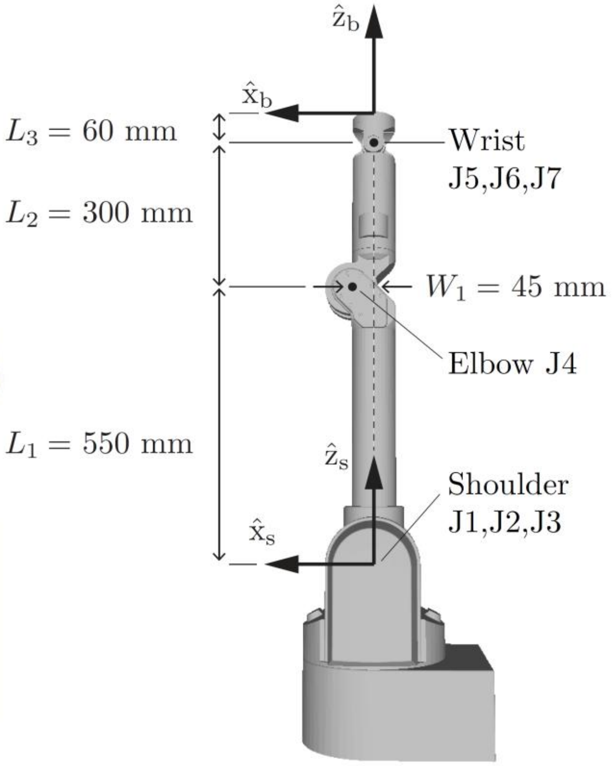
i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,0)$
4	$(-1,0,0)$	$(0,0,L)$
5	$(-1,0,0)$	$(0,0,2L)$
6	$(0,1,0)$	$(0,0,0)$

Space form

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

Body form

Screw Axes in the End-Effector Frame



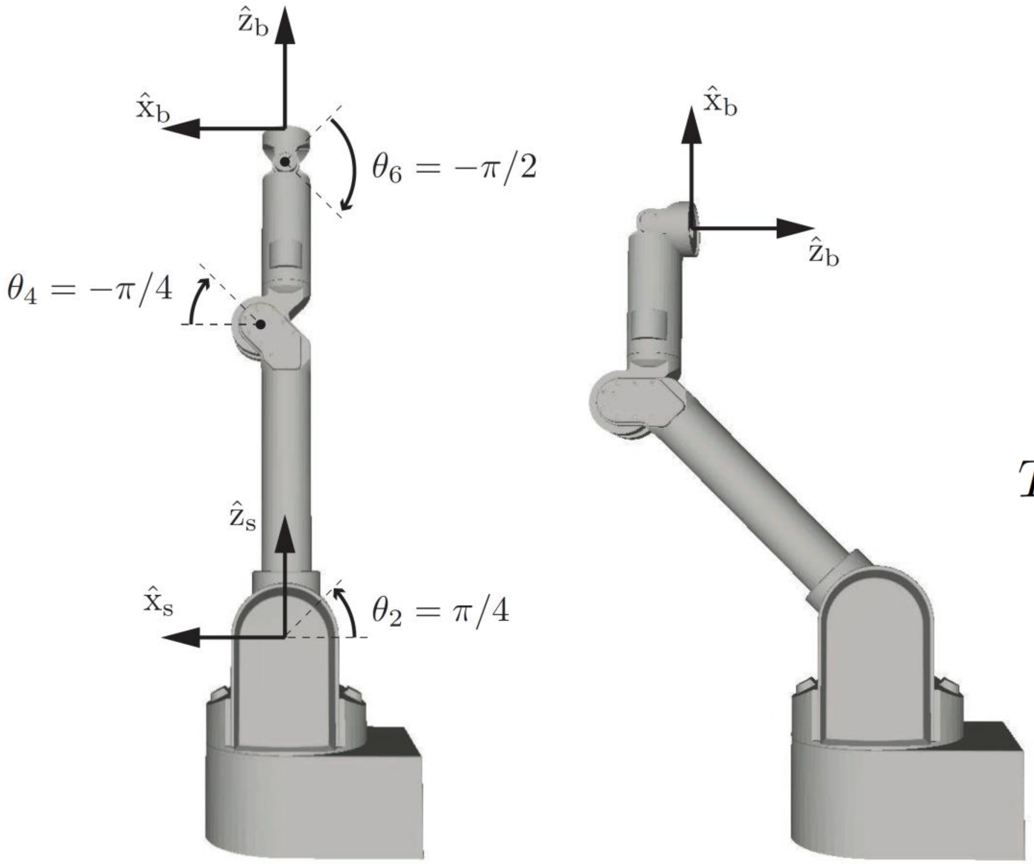
Barrett Technology's WAM 7R robot arm at its zero configuration

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_i = (\omega_i, v_i)$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(L_1+L_2+L_3,0,0)$
3	$(0,0,1)$	$(0,0,0)$
4	$(0,1,0)$	$(L_2+L_3,0,W_1)$
5	$(0,0,1)$	$(0,0,0)$
6	$(0,1,0)$	$(L_3,0,0)$
7	$(0,0,1)$	$(0,0,0)$

Screw Axes in the End-Effector Frame



Barrett Technology's WAM 7R robot arm at its zero configuration

$$\theta_2 = 45^\circ,\ \theta_4 = -45^\circ,\ \theta_6 = -90^\circ$$

$$T(\theta) = Me^{[\mathcal{B}_2]\pi/4}e^{-[\mathcal{B}_4]\pi/4}e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

- Screw axis

- Product of Exponentials Formula Spatial twists
  - Spatial form

  - Body form

Further Reading

- Chapter 3 and Chapter 4 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017